SEQUENCES, THEIR APPLICATION AND USE IN THE FIELD OF MATHEMATICS

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Introduction

As my interest in the field of mathematics has grown and offered many possible career options for later in life, so has my need to learn more about mathematics than simple calculations. Previously lacking ambition to pursue a possible career in mathematics has led to an absence in commitment to a specific career field. This year I established my strengths, consisting mainly of logical and quick thinking as well as the ability to process and perform mental calculations at a fast rate. Taking this into consideration, I determined that I should commit to the field of mathematics. This turned out to be a simple decision as mathematics and physics had already been my strong points and the subjects I enjoyed the most.

My personal project this corresponds to these interests and the deductions I made. When deciding what topic, I should choose to precisely analyse, my thought process was the following. I was open to a lot of different areas; however, my main objective was not only to be able to work for myself but also to be able to convey some of the acquired knowledge to my classmates who might not have discovered the beauty of mathematics. With this said, I wanted to choose an area which seems simple but is actually complex and intricate when closely analysed. This, in theory would aid other students into thinking more about things that appear simple.

After that, the answer came imminent and was clear, ‘mathematical sequences’ was the subject I decided upon to undertake at the start of the year, later this was then extended to ‘mathematical sequences, their application and use in the field of mathematics’ as the previous title was rather vague and did not truly express what would be written about. The subject appeared apt to me as it corresponds to my interests and main objective that I had previously specified, other reasons included having come across famous sequences, such as the Fibonacci sequence, but not having fully comprehended the notion.

This work will include a basic introduction to the notion, history, properties and notation of sequences in general, this part of the project will be clear to someone who has not studied the subject beforehand. This section also serves the purpose of aiding people understand the rest of the work, which consists of more complex analysis of various sequences. Furthermore, the applications and uses of different mathematical sequences in and out of the field of mathematics will be included to enlighten readers about the fact that sequences are not just lists of numbers in mathematics, but that they play a role in day to day life in many different areas such as science and photography.

After the explanations, three mathematical sequences, hand selected by me, have been included. Here, the sequences will be analysed in full detail, with applications and properties looked at more closely. These sequences have been chosen, as they stood out to me whilst conducting research for the rest of the project. This is because of the beauty, complexity and recognition of each sequence, some will contain a controversial piece of history, or an immortality matched by no other sequence.

After having completed this project I hope to firstly attain a greater understanding of not only sequences and sequencing but the whole field of mathematics as a result of the research done. The main objective to be able to covey this information to my classmates will most likely not be possible due to the coronavirus, I however will be happy to show this work to anyone who expresses interest in the subject. However, something that I also wish to attain is a finalised document, which fully explains what a sequence is made up of and each different component explained in full detail.

What is a sequence

From Latin 'sequens' meaning 'following' a sequence is a collection of objects like numbers, letters and symbols, these can be listed in any way possible irrespective of repetitions and order. Each sequence has a specific length which is defined by the number of elements in the sequence. The length can be either finite (meaning there are a set number of elements in the sequence) or infinite (meaning there is no end to the sequence) an example for these are $\pi = (3.141592653 \ldots)$.

So, for example in the following finite sequence:

$\{1, 2, 3, 4\}$ The length of the sequence is 4, as the sequence contains 4 elements.

Sequences are useful in a number of mathematical disciplines for studying functions, spaces, and other mathematical structures.

In particular, sequences are the basis for series, which are important in differential equations and analysis. However, sequences are also used in science and computing, in science DNA sequencing has greatly contributed to biological and medical research and discovery. In computing for example, a sequence is used in an SQL database, to generate a unique number and starting point, which is then increased based on a set interval.

The difference between a set and a sequence

The element in a sequence can be placed in any which way, the elements can appear twice and at different positions in the sequence. The sequence can repeat itself and order does not matter.

However, in mathematics, a set is a defined collection of objects[1], numbers are distinct objects when seen separately. However, if seen collectively the three numbers can be seen as set of a certain size. For example:

$\{4, 8, 12\}$

These numbers have been seen collectively and they now form a set of size three. This theory is important in mathematics and the set theory was developed in the 19th century. So important in fact that the set theory can be seen as the foundation of nearly all mathematics. Given that the first counting/mathematics probably started with these collections of defined objects.

A set has some basic properties including that a set contains objects, and that two sets are equal if every object is also an object of the other set.

[1] Also called an element, but object is the name more commonly used when referring to a set.

Notation

Each element has a rank (also known as index) and this rank defines the position of the element. Following this the first element of each sequences either has rank of 0 or 1 depending on the sequence.

A typical notation of a sequence would appear like this:

$\{a_n\} = \{1, 4, 9, 16\}$

The sequence starts with the letter 'a' (lower case) in braces which expresses the name or the identification of the sequence. A sequence can also be denoted by one of the elements in the sequence. In that case the sequence will be named after the rank of that specific element.

Followed by the identification the subscripted letter 'i' or 'n' which can have two meanings based on different notations with and without braces.

The notation without braces being the 'index' or the counter for example 'a_2' which simply designates the specific term in the sequence, in this case the second term in that sequence. However, the notation with braces which then refers to the entire sequence. (Either curly brackets or standard brackets are used when noting sequences and both have the same meaning.)

The equals sign then clearly shows what the sequence is and then followed by the actual contents of the sequence.

The notation can also define the starting term and the ending term at the time. For example, if the starting term in the sequence were to be 1 and the last term n, it would be written as:

$\{a_i\}^n{}_{i=1}$

Another example being, if a sequence started with the index 3 and going on forever would be written as:

$\{a_i\}^\infty{}_{i=3}$

This sequence has not got a defined ending value as it is infinity, and there for is classed as an infinite sequence. Note that most infinite sequences have a finite lower index.

The index letter i, indicating the beginning value of the counter is called the lower index, and the ending value outside of the brackets written in superscript is known as the upper index. This is logical due to their position in the notation. The lower index can even be zero.

Contrary to sets (see page 5) sequences cannot be shortened or simplified.

Therefore, the following sequence:

$\{a_n\} = \{1, 2, 1, 2, 1, 2, 1, 2\}$

cannot be rearranged or "simplified" in any manner.

Indexing (Rule)

Sequences often contain a rule which would determine the next term of a sequence. Each succeeding term is determined by the same rule. A rule is a mathematical process of calculating the next term, whilst using the same calculation. A sequence only contains a rule if the terms of the sequence are in a non-random order, therefore allowing for the next term to be identified. The rule of a sequence is often contained within the notation of the sequence. Often this rule is related to the index. So, for instance, in the sequence:

$A = \{a_i\} = \{2i + 1\}$

The i^{th} term is defined by the rule $2i + 1$ the terms would be ordered like this:

$a_1 = 2(1) + 1 = 3$

$a_2 = 2(2) + 1 = 5$

$a_3 = 2(3) + 1 = 7$

The rule being: multiplying the previous term in the sequence by two and then adding one, and this process then calculates the next term in the sequence.

There are different types of sequences that are determined by a rule or index. The nature of these sequences is determined by the type of repetition that determines the sequence's next element. If the index of the sequence is to either add or subtract from the previous number, then the sequence is called an arithmetic sequence. However, if the index of the sequence is to multiply or devise, then the sequence is called a geometric sequence. [2]

[2] Geometric and arithmetic sequences in detail pages: 8 – 9

Recursion

A recursion is another method to determine the elements of a sequence, another slightly altered index which uses the previous elements of the sequence to establish the following elements. This method is however only applicable to sequences whose elements are related to the previous elements. These sequences are denoted as recursive sequences whose index is called a recursion.

To denote a sequence by recursion, a rule is necessary, called recurrence relation to establish each element in terms of the previous elements. However, a certain number of initial elements need to be defined so that all succeeding elements of the sequence can be determined by the application of the recurrence relation. The most apt and possibly most famous example of a recursive sequence is the Fibonacci sequence. Here the recurrence relation is much simpler than the actual index of the sequence.

For example, the Fibonacci[3] sequence [arithmetic sequence] which pursues the following recursion:

$a_n = a_{n-1} + a_{n-2}$

- a_n being the element number "n"
- a_{n-1} being the previous element
- a_{n-2} being the element before that

This can be proven as follows:

When determining the 6^{th}[4] element of the sequence, the two preceding elements are three and five.

$$a_6 = 5 + 3 = 8$$

[3] $F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

[4] The indexing of a sequence commences at index 0 not index 1. Therefore, the 6th element is the 7th number in the sequence.

Geometric and arithmetic sequences

There are two sequences that dominate the scene when it comes to the application of sequences, due to the index of both. The index consists of either a division or multiplication and an addition or a subtraction of the previous elements.

This phenomenon appears often when calculating budgets and investments, as these usually increase in a structured manner. Allowing for the sequence of an investment for example, which needs to be calculated in advance by adding or multiplying the previous investments.

Geometric sequences

As already seen before hand, a geometric sequence, also known as a geometric progression, is a sequence where each element (usually succeeding) is determined by multiplying or dividing the previous elements by a fixed non-zero number. This number is known as the common ratio. Apart from multiplications and divisions, powers can also be the common ratio of a geometric sequence. These powers are represented as follows: $r^d \; or \; 2^d$. Typically, 'r' is any non-zero number and the sequence denoted 'a' is the scale factor equal to the starting value of the sequence [see below].

Examples

- For the sequence $(a_n) = \{1, 7, 49, 343, \dots\}$ the common ratio is 7, because each preceding element is always multiplied by 7 to determine the next element of the sequence.

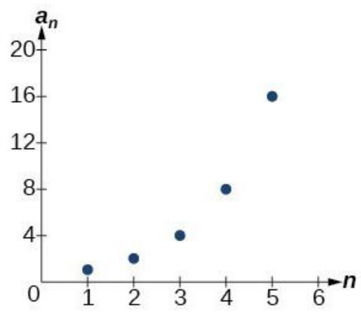
- The sequence $(a_n) = \{a, ar, \dots, ar^d, ar^{d+1}, ar^{d+2}, \dots\}$ is an example of a geometric sequence where the common ratio of the sequence is a power.

Properties

- Given that the index of all geometric sequences is based upon the preceding terms of the sequence all geometric sequences follow a recursive relation.

- To prove the validity of a geometric sequence, one would have to find a common ratio that repeats itself across all elements.

- If the index repeats itself on all elements the sequence becomes infinite and grows exponentially large, only if the common ratio is ≥ 1 and doesn't consist of a division.



Example of a geometric sequence growing exponentially large

Arithmetic sequence

Arithmetic sequences, also known as arithmetic progressions, are similar to geometric sequences, due to the similarities in properties. The elements of an arithmetic sequence are, like geometric sequences, dependent upon the preceding elements of the sequence. However, contrary to geometric sequences, the elements are determined by adding and subtracting a certain number from the previous elements of the sequence. All arithmetic sequences increase or decrease in a linear manner, meaning constant, observable in the graph below.

This number remains the same throughout the sequence and is known as the common difference. The common difference of an arithmetic sequence is determined by subtracting the first of two sequential elements from the second, therefore the result is denoted 'difference'. Within a sequence the common difference is denoted as 'd'.

<u>Examples</u>

- For the sequence $(a_n) = \{23, 6, -11, -28, -45, -62, \ldots\}$ the common difference is – 17, because to determine the next element, one would need to subtract 17 from the previous element.

- The notation of the index of the sequence above would be as follows: $(a_n) = a_1 + (n-1)d$, this can be proven by for example calculating the third element of the sequence from above like this:

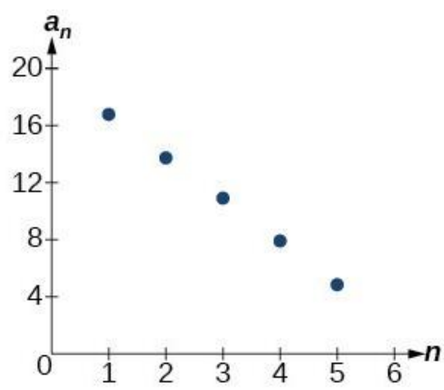
$$23 + (3-1) * (-17) = -11$$

Properties

- The nature of an arithmetic sequence depends on the common difference.

- If the common difference is positive, the elements will go on to positive infinity.

- If the common difference is negative, the elements will go on to negative infinity



Example of an arithmetic sequence with a negative common difference. Visible are the numbers decreasing in a linear (constant) manner.

Properties of sequences

Increasing and decreasing

A sequence where the next element is always greater in size or the equal to the previous sequence is called a monotonically increasing sequence. Being more specific still, a sequence where the next element is only greater than the previous element is called a strictly monotonically increasing sequence. The other extreme where the succeeding element is less than the previous element is logically called a monotonically decreasing sequence. If there is no specific order to the elements of the sequence phrases such as nondecreasing and nonincreasing are applicable.

Examples:

1) Monotonically increasing sequence: $(a_n)_{n=1}\ only\ if\ a_{n+1} \geq a_n$
2) Monotonically decreasing sequence: $(a_n)_{n=1}\ only\ if\ a_{n+1} \leq a_n$

Bounded Sequence

A sequence of numbers is either bounded from above or below, this occurs when all of the elements are either smaller or greater to a real number T. For example, a sequence is bounded below if all of the elements greater than or equal to the real number T. A sequence is then also bounded above if the opposite happens (all elements are either equal or smaller to the real number T).

Examples:

1) Bounded from below: $a_n \geq T\ (T = real\ number)$
2) Bounded from below: $a_4 = 4\ and\ real\ number = 2\ then\ 0 < 2 < 4$
3) Bounded from above: $a_n \leq T\ (T = real\ number)$
4) Bounded from above: $a_n = 2\ and\ real\ number = 3\ then\ 0 < 2 < 3$

Finite and infinite sequences

Finite sequences are simpler; the sequences have a set number of elements and the length is usually defined by n. However, infinite sequences can be either be infinite in one direction or the other, meaning that the sequences have a set first element but no set final element. These sequences are called singly infinite sequences. As previously stated there is another type of infinite sequence where the sequence is finite in both directions, this means the sequence has no set starting or finishing element. This type of sequence is called a two-way infinite sequence.

Examples:

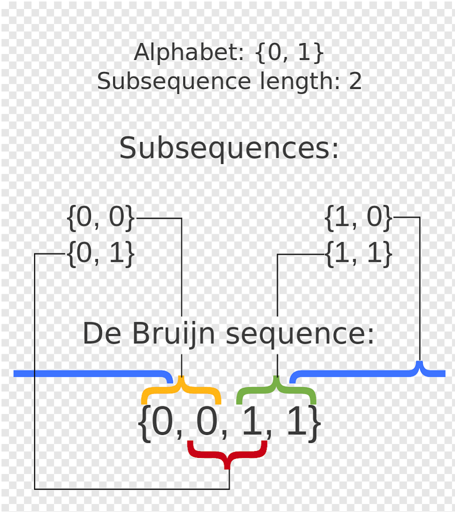
1) Finite sequence: $\{a_n\} = \{1, 4, 9, 16\}$
2) Singly infinite sequence: $\{1, 2, 3, \dots\}$
3) Two-way infinite sequence: $\{4_n\}\ n =_{-\infty}^{\infty}$

Subsequences

A sequence may also have a subsequence, meaning that the subsequence was created by deleting elements from the original sequence, whilst keeping the relative positions of the remaining elements. No elements are added; this means that all the elements in the subsequence are also contained within the first sequence.

Sequences are difficult to analyse because the complexity and the amount of data, subsequences are subsequently created as they are simpler and provide a useful tool for understanding them. A great deal about the original sequence can be determined by analysing the subsequence. Given that all elements of the subsequence can be found in the original sequence, they share a lot of each other's properties and are therefore closely tied together.

For example, the sequence (2, 4, 6, …) is a subsequence of the sequence (1, 2, 3, …). The positions of some of the elements have changed, however the positions of the original elements are kept.



Subsequences of the De Bruijn sequence

Converging sequences and their limit

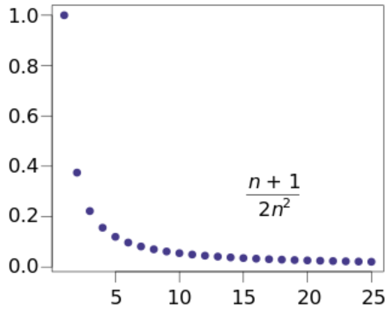
If a sequence converges it means the sequence will merge into a specific value which is called the limit of the sequence. Therefore, a sequence that converges is known as a convergent sequence. Logically every other sequence which does not converge is called divergent.

The elements of a convergent sequence become closer to their set limit, meaning that the elements will never exceed this certain limit. A common example [from Wikipedia] of a divergent sequence is:

$a_n = \frac{n+1}{2n^2}$

With the help of a graph it is easily visible that this sequence merges into the value 0. This means that the limit of the sequence:

$a_n = \frac{n+1}{2n*2n}$ is 0.



As n (number of elements) increases the sequence converges.

Divergent sequences are for example singly infinite sequences, these are divergent due to the fact that the elements are never ending, however the difference between the first convergent sequence is that these sequences do not approach a certain limit.

Example:

$A_n = \{n^1, n^2, n^3, \ldots\}$

Infinite convergence

A convergent sequence can also converge to infinity, here the elements of the sequence merely grow indefinitely large and therefore grow to either negative or positive infinity, denoted as follows:

$$\lim_{n\to\infty} a_n = \infty$$

Here as n grows infinitely close to positive or negative infinity, the limit for a_n becomes infinity as it is clear that the terms will never surpass infinity. Even though this sequence is divergent it is said to converge to positive infinity. The same can also be denoted for a divergent sequence that converges to negative infinity:

$$\lim_{n\to-\infty} a_n = -\infty$$

Applications of sequences

Sequences and series are very important in mathematics, where their uses are spread out across areas such as finance, statistics and physics.

In mathematics sequences are used for studying functions spaces and structures using the convergence of sequences as mentioned before. However, they are also in analysis (finance) and equations.

In the field of mathematics

Finance

- *Investments*

A comprehendible example is if I invest 600€ in a commodity at an 8% interest rate, I will be able to calculate the investment made each year. This works by being able to calculate 8% percent onto the previous investment. For example, if I start out with an investment of £600 the investment will continue like this:

$\{600€, 648€, 699.84€, 755.83€, 816.29€, 881.60€, 952.12€, 1028.29€\}$

(Each element of the sequence representing the amount invested every year [1-8] rounded to two decimals.)

This progression is a sequence, and it is also possible to calculate the recursion of the sequence. This calculation is often used in finance to save time and spare unnecessary calculations. The next element of the sequence can always be determined using the recurrence relation. In this case it is quite simple, using only two values: the amount invested (q) and the interest rate (r). Then an interval (time period between each r%) which is repeated n times. So, then the next element would be determined by the following recurrence relation:

$$(a_n) = a_{n-1} * \left(\frac{r}{100}\right) + a_{n-1}$$

- *Depreciation*

Depreciation is another circumstance closely linked to investments where sequencing is used to determine each element of a progression. Depreciation is the constant loss of value of an object over a set period of time. This phenomenon is usually represented by a percentage, this means that the phenomenon is the same as investing, the only difference being the fact that an amount is always subtracted from the total compared to added. For example, a mobile phone that is bought for £400 depreciates at a rate of 10% for the first year and every year the rate drops by 1%. N represents the previous element of the sequence and 'I' represents the year. The elements of the sequence would be ordered as follows:

$\{£360, £327.6, £301.39, £280.29, £263.48\}$

[Year 1-6]

Here the recursive rule would be as follows:

$$(a_n) = a_{n-1} - \left(\frac{10-i}{100} * a_{n-1}\right)$$

Convergence and limits

The limits of a sequence as well as its convergence are extremely important when it comes to mathematical analysis. Mathematical analysis is the field of mathematics which handles the limits of sequences, which are the core on which the whole field is based upon. The application of sequences is equal to the whole field; therefore, the application here is to determine the limit of a sequence. Clever calculations allow this to be possible. The following are the rules of calculation of convergent sequences and their limit.

The following limits are true if (a_n) and (b_n) are both convergent sequences and c representing all real numbers:

- If the limit of (a_n) and (b_n) added to or subtracted from each other is infinity, then the result is equal to the addition and subtraction of both limits, noted like this:

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

- The limit of (a_n) multiplied by c, is equal to c multiplied by the limit of (a_n) noted like this:

$$\lim_{n\to\infty} c * a_n = c * \lim_{n\to\infty} a_n$$

- The limit of (a_n) times (b_n), is equal to the limit of (a_n) multiplied by the limit of (b_n), noted as follows:

$$\lim_{n\to\infty} (a_n * b_n) = (\lim_{n\to\infty} a_n) * (\lim_{n\to\infty} b_n)$$

- If $b_n \neq 0$, then the limit of (a_n) divided by (b_n) is equal to the limit of (a_n) divided by the limit of (b_n), noted like this:

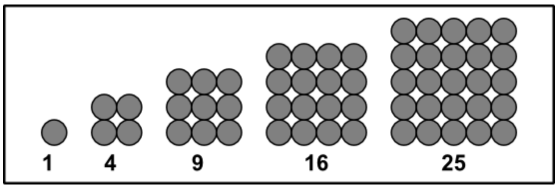
$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

- If $p > 0 \; and \; a_n > 0$, then the limit of (a_n to the power of p) is equal to the limit of a_n, to the power of p, noted as follows:

$$\lim_{n\to\infty} (a_n^p) = (\lim_{n\to\infty} a_n)^{\,p}$$

Exponential growth

The expansion rate of something can easily be predicted or foreseen, after having applied this method. This scenario is very common and often represented with objects such as the square numbers sequence. This can be used to predict population increase and the spread of disease among others. As is often the case, the index needs to be determined so that each other element of the then created sequence can be established.



Example of a figurate number system, here the figurate number system of a square

In these cases, the most used mathematical phenomena used are figurate number systems, which are members of different sets of numbers. This simplifies the spread or growth of a certain object.

<u>Examples</u>

- A deadly disease is being spread and scientists need to estimate the risk and how many people could be affected by the disease. If an inffected patient gives the disease to 5 people every week and these people, then in turn also pass the disease on to five other people the next week each. How many people would be infected within a month? N represents the number of weeks and 'I' represents the previous element of the sequence.

  {1, 6, 31, 156, 781} [people infected each week]

  $(a_n) = 5^{n-1} + a_{i-1}$

  After five weeks the number of people infected would be $5^7 + 19\ 531 = 97\ 656$.

- The population's increase

- The expansion growth of the internet/date

Other uses

These are only a few of the different uses for sequences in the field mathematics, there are other uses less known to people who are not familiar with the exact field of mathematics. Another well-known example would be analysis, where sequences are also used, however most other applications are more complex and less common for example topology.

Topology is a field in mathematics that deals with geometrical properties and the relation of different spaces, not dependent on the change of its size or shape. Here the sequences play an important role, belonging mostly to the study of metric spaces.

Applications of sequences outside of mathematics

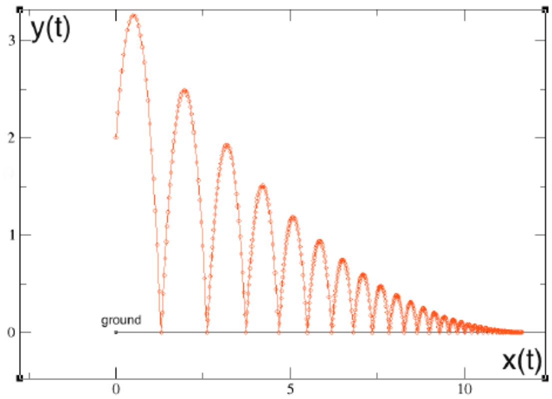
Sequences are not only a concept used within the field of mathematics, in fact everyone uses sequences irrespective of whether thy use mathematics on a daily basis.

These are the uses of sequences in real life, the everyday applications of sequences that you implement without thinking twice about, usually without any calculations.

Movement

The first example of the use of sequences in daily is in fact on of the most common performed processes that we complete. We do this without much thought and usually are able to complete the action. The action is carried out across many sports such as basketball, football and many others.

The action is simple: bounce a ball.



Trajectory of a ball bounce on a flat surface, clearly observable is the sequenced energy loss of the ball.

When bouncing a ball, it bounces at a certain rate and then logically loses this energy over a certain time period. This process has been studied and the conclusion was that the ball loses its energy at a set rate, with the same fraction of energy being deducted every time the ball bounces. And during a basketball game the players factor this in without even thinking about it. The same applies if you are playing football with friends and you take a free kick, you would need the factor in how much the ball drops every second, which is something we all do.

This is something we all use in our daily lives, this example may be without any calculations, but it still applies, everyone uses sequencing to predict the trajectory of objects.

DNA Sequencing

DNA sequencing is used to determine the nucleic acid sequence, used in the field of science. This information is of vital importance for a researcher in understanding the type of genetic information that is contained within our DNA. It specifies the order in which the constituents find themselves in, which could lead to the premature detection of certain genetic illness and may even foresee and prevent illness occurring in the first place. Here, the notion and of sequences is used to help save lives.

Other uses

Most of society and reality around us is based upon sequence after sequence, changing and repeating themselves over and over again. Common example of this are time and the calendrical system. Time [seconds, minutes, hours] always follow the same sequence, which always contains the same number of elements, in this case either 24 or 60. Likewise, the church bells, we must first figure out the rule of the sequence of 'dings' before we know what time it is. The same principle is also valid for the calendrical system.

Our lives are ruled over by sequences such as the routines that we follow every day without knowing, leading to their great importance in the structure and function of the modern world.

Famous mathematical sequences

Fibonacci sequence

Introduction

This sequence was certainly the obvious choice, given that it is one of the most famous sequences in history, most people would have already heard of the sequence. However, there are many more aspects to the sequence than most people think. The sequence has a long history and has a strong connection with mathematics, it is also strongly connected to nature and other phenomena, making the sequence extremely interesting to study.

The Fibonacci sequence is a sequence of real numbers starting at zero, where the next element is determined by the sum of both preceding elements.

Notation

The Fibonacci numbers are commonly denoted as follows:

$$F_n$$

The single elements of the sequence would be, for example, denoted as follows:

$$\{F_0 = 0\}\,\{F_3 = 2\}$$

These then create the Fibonacci sequence, which in itself is a singly infinite sequence meaning it has a set starting number [0] but does not have a set ending number. Given the index of the sequence it grows exponentially large very quickly. All together the first ten terms of the Fibonacci sequence would be denoted as follows:

$$(F_n) = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

The Fibonacci sequence evidently follows a simple recurrence relation by the way the succeeding elements are defined. This straightforward relation can be determined in the following manner:

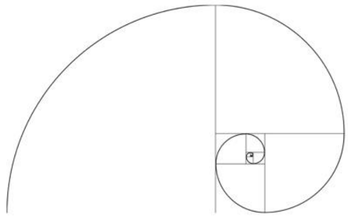
$$F_n = F_{n-1} + F_{n-2}$$

(This recurrence relation is however only valid for $n > 1$.)

Despite the simple recurrence relation, the rule of the Fibonacci sequence is considerably more complicated. The index of the sequence can only be determined by using a first-order linear recurrence. This when a variable at a certain time, is dependent on its value at previous times, here however the value of a first order recurrence only depends on one specific point of time. After completing a long range of calculations, the index of sequence can be denoted in the following manner:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

History



An example of the iconic Fibonacci spiral

The true origin of the Fibonacci sequence cannot be determined, however the earliest known knowledge of the sequence come from around 2 500 years ago, about 450 BC. The information tells us that Indian mathematician Pingala first came up with the notion of the sequence. Meaning that Fibonacci did not discover, nor invent the sequence in any way contrary to common belief. According to various sources, many other mathematicians are credited with the discovery of sequence, for example the Indian polymath Hemachandra in 1150 AD.

The first mention of the sequence in accordance with Fibonacci appeared in his book Liber Abaci, which appeared around 1202. Even then, the name Fibonacci was only first associated with the sequence in the 19^{th} century.

Visible here is an outtake of Fibonacci's famous book – Liber Abaci

The way Fibonacci used the sequence was to calculate the number of rabbit offspring each year. Here Fibonacci assumed the growth of the rabbit population, by considering that a pair of new-born rabbits are placed into a field, where the pair start to breed after a period of one month. Then, Fibonacci supposed that after each month the pair of rabbits produce another pair of rabbits ready to breed in another month. Fibonacci then asked the question, presuming that the rabbits never die but continue breeding forever, how many pairs of rabbits there will be after one year. This problem is easily solved with the Fibonacci sequence.

However, Fibonacci himself was not very interested in the sequence and any of its properties at all. In a controversial piece of history, apart from the small outtake of his book [Liber Abaci] about the rabbit breeding issue, Fibonacci never mentioned the sequence nor its properties again. In fact, Fibonacci never even named, the sequence after himself or at all.

After the small and almost useless introduction to the sequence by Fibonacci, it was forgotten for around 600 years. During this period no one worked on the sequence neither did anyone show any particular interest in the sequence. Then, after a long pause in the history of the sequence, it reappeared in the late 19^{th} century. Mathematicians found the sequence interesting due to its many complex mathematical properties; this led the team of mathematicians to analyse the sequence in depth. In 1877, the French mathematician Édouard Lucas officially named the sequence 'the Fibonacci sequence' due to his mentioning of the rabbit problem 600 years earlier.

Negative Fibonacci numbers

The Fibonacci sequence which many imagine to only have positive numbers [including 0] is also able to be negative. Under the term negative, we understand negative Fibonacci numbers, meaning the sequence can contain negative elements. However, we also understand negative indices[5] under the term negative, meaning that a negative 'n' exists.

The negative elements of the Fibonacci sequence, also called 'negafibonacci' numbers, can be denoted in the following manner:

$$F_{-1} = 1, F_{-2} = -1, F_{-3} = 2, F_{-4} = -3, \ldots$$

As can be seen above, the actual negative elements are spaced out, meaning that a negafibonacci number, where the element is actually negative, is always followed by a positive negafibonacci number. The nature of the negafibonacci number is determinable through a slightly altered recurrence relation:

$$F_{n-2} = F_n - F_{n-1}$$

This can be proven for example by calculating the 6th negafibonacci number as follows:

$$-3 - 5 = -8$$

[5] Plural of index

Applications of the Fibonacci sequence

Here is a list of a few of the uses of the Fibonacci sequence, the domain will be stated, followed by a short explanation about the use of the sequence.

- *Technical analysis*

A method used to determine support and resistance levels, known as Fibonacci retracement, is applied for financial marketing and trading.

- *Algorithms*

A few pseudorandom number generators[6] use 'random' elements of the Fibonacci sequence to create the pseudorandom numbers, which in turn have a variety of different uses, such as electronic games.

- *Unit conversion*

The conversion rate for miles to kilometres (1.609344) is very close to the golden ratio, therefore a decomposed number of miles into the sum of the kilometres is very close to the sum of the Fibonacci numbers if they are displaced by the preceding numbers.

- *Computer science*

Here the Fibonacci numbers appear in the analysis of the Fibonacci heap, which consists of a data structure for priority queue operations. Another use is for a one-dimensional optimization method, which is used as a search function in programming.

- *Mathematics*

The Fibonacci numbers play a part in the computer-generated algorithm to determine the greatest common divisor of two different numbers. This algorithm was created by Greek mathematician Euclid. The property that the Fibonacci numbers possess here is that each worst scenario input for the algorithm are two succeeding elements of the Fibonacci sequence.

- *Agriculture*

As already seen the Fibonacci sequence can be used to calculate animal as well as human populations by considering the amount of breeding pairs and their breeding rate of the certain species.

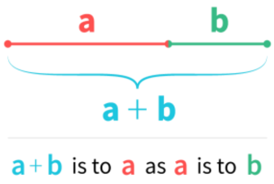
Other uses

The various applications of the Fibonacci sequence and Fibonacci numbers are not limited to the list above. Many other applications exist in different domains.

[6] These are algorithms for generating a sequence of numbers whose properties approximate the properties of random numbers

Relation to the golden ratio

The golden ratio often represented as phi the 21st letter of the modern Greek alphabet (ϕ) is an indefinite and irrational number. This means that it consists of an unending amount of terms after the decimal point, phi is often approximated to 1.618. The exact value is $\frac{1+\sqrt{5}}{2}$.



In terms of mathematics two amounts are in the golden ratio, if the ratio of both quantities is equal to the ratio of their sum to the larger amount. This phenomenon is only valid for two quantities where a > b > 0. This can also be expressed mathematically in a more conclusive and detailed manner. For example, as follows:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi$$

This means that if the larger length (a) added to the smaller length (b) is equal to the ratio between both lengths, then they are in the golden ratio (≈ 1.618). A rectangle whose sides are in the golden ratio, is said to be the most beautiful rectangle, the most eye pleasing rectangle. Many mathematicians have researched this occurrence starting with Greek mathematician Euclid, who also studied the properties of the Fibonacci sequence.

The reason the Fibonacci sequence has been linked with the golden ratio is because when finding the ratio of two Fibonacci numbers, specifically the ratio between one element and the previous element. Here it is clearly possible to see that each division is extremely close to the golden ratio (≈ 1.618). This can be proven with a few examples:

$$\frac{89}{55} = 1.618618 \ldots \ and \ \frac{46368}{28657} = 1{,}618033 \ldots$$

However, the real reason the Fibonacci sequence and the golden ratio are so closely intertwined is due to the 'index' of the Fibonacci sequence which contains phi. The index of the sequence can also be expressed as follows:

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

The golden ratio is equal to the limit of the division between successive Fibonacci numbers, meaning that as n becomes larger the divisions merge towards the golden ratio. This property can be denoted as follows:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

Expressed differently, a Fibonacci number divided by its preceding element, the result is approximately equal to the golden ratio. The result is either lower or higher than the golden ratio and as n increases the ratio converges to the golden ratio. Denoted more generally:

$$\lim_{n\to\infty} \frac{F_{n+a}}{F_n} = \varphi^a$$

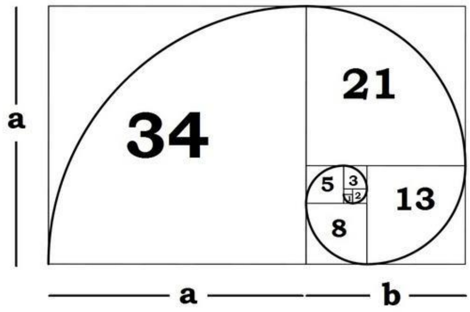
Furthermore, the powers of ϕ obey the recurrence relation of the Fibonacci sequence:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

The Fibonacci sequence in photography

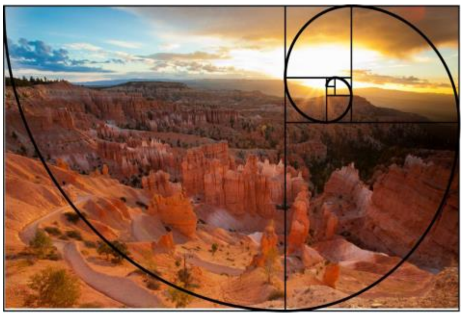
Amongst the various uses and applications of the Fibonacci sequence and numbers, photography holds one of the strongest connections with both the Fibonacci sequence and the golden ratio. This is because the golden ratio assists in taking attractive photos with a strong composition. The golden ratio balances out the image, meaning that if a photo is taken with the golden ratio it will seem more eye pleasing to the viewer.

This way of taking photos and even painting has been used for hundreds of years, paintings such as the Mona Lisa and the last supper are rumoured to have been painted with the golden ratio. The main aim of photography is to create visually pleasing images, to which the golden ratio is the answer.



The Fibonacci spiral is at heart of the well composed pictures, this spiral is created whilst dividing the elements of the sequence by their immediate predecessor. Thanks to the image directly adjacent, it is possible to determine that all rectangles are in the golden ratio, thus providing conclusive evidence for the relation between the sequence and the golden ratio.

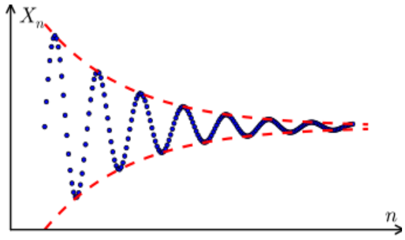
This spiral is a composition guide, it guides the eye through the entire photo, leading to more captivating images. The golden spiral, as applied to photography, suggests placing the subject on the smallest box in that spiral. Placing other prominent areas of the image on the remaining curve, wherever possible, will lead the eye of the viewer through the image.



Cauchy sequences

Introduction

The Cauchy sequence [pronounced: Co-she] is a mathematical sequence where all the elements of the sequence become extremely close to one another as n gets larger. Because of this property the sequence is often used as an example to demonstrate the convergence and limits of a sequence. This is due to the fact that as the sequence progresses all elements convert to a given number. This number can vary given that the word Cauchy is given to an infinite number of sequences unlike the Fibonacci sequence, whose elements are predetermined. A Cauchy sequence can consist of both real and complex numbers.



Graphical representation of a Cauchy sequence

Notation

A mathematical sequence is given the name Cauchy when two requirements are fulfilled. The first of these is there must be a positive integer, denoted as N, for all the positive elements of the sequence. These positive elements are all real numbers and are denoted as ε. The second requirement is that

$$n > N$$

The ‘n’ stands for natural numbers of the sequence, to conclude, all the natural numbers must be larger than the positive integers of the sequence. However, it does not suffice for each element to become arbitrarily close the previous term.

Accordingly:

$$|x_m - x_n| < \varepsilon$$

Here the vertical bars on either side of the equation represent the non-negative value of x also called the absolute value. Meaning that the absolute value of the difference of two consecutive elements must be smaller than the positive elements. Here the difference must be unmeasurable for every pair of m and n.

History

When determining if a mathematical sequence is convergent, a predetermined limit needs to be utilised before one could test the definition. This was a common problem, for which Bernard Bolzano first discovered a solution. Mr Bolzano solved the problem using an idea previously introduced in the early 19th century. This concept, later developed by Bernard Bolzano, was the idea of French mathematician, engineer and physicist Augustin Louis Cauchy (1789 – 1857), who made several vital contributions to many branches of mathematics, including mathematical analysis and continuum mechanics.

Properties of Cauchy sequences

- *Convergence*

Every convergent sequence with a predefined limit is Cauchy and therefore a Cauchy sequence. This can be proven, given any real number greater than 0 (ε), beyond a certain point each term is spaced out by less than $\frac{\varepsilon}{2}$ of the limit of the sequence. This concludes that any two elements of the sequence are within ε of each other. Due to this property it can be said that a sequence of real numbers is only convergent if it is also Cauchy. However, there are sequences [that are Cauchy] of rational numbers that are not convergent in the rational numbers. Meaning that these sequences have no rational limit. Summarised, any sequence of rational numbers that converges to an irrational number is Cauchy, is however not convergent when seen as a sequence in the set of rational numbers.

Bounded Cauchy

When referring to a Cauchy sequence [x_n] in a metric space, it is invariably bounded[7]. This is the case as all elements from a defined point onwards, denoted as N, are distanced within 1 of each other, then if M is the greatest distance between x_N and all other terms up until N, thus no element of the sequence is spaced out by more than M+1 from x_N.

Subsequences of a Cauchy sequence

When referring to a Cauchy sequence [x_n] in a metric space with a convergent subsequence with a limit is itself also convergent, if the subsequence yields the same limit as the original Cauchy sequence. Proving this is simple, any rational number [r] greater than 0, beyond a predetermined point in the original Cauchy sequence, every element included within the subsequence is spaced out by a distance smaller than $\frac{r}{2}$ of the limit of the sequence. This satisfies that any two elements of the original sequence are within $\frac{r}{2}$ of each other, meaning every term of the original sequence is within distance r of the limit.

[7] The definition of bounded sequences can be found on page 10

Applications of Cauchy sequences

Here is a list of a few of the uses of Cauchy sequences, the domain will be stated, followed by a short explanation about the use of the sequence.

The uses of Cauchy sequences are generally limited to mathematics and other domains based around mathematical principals. This is unlike the Fibonacci sequence which is a famous mainstream sequence, Cauchy sequences on the other hand are sequences strictly limited to mathematical properties and are therefore more focused on that domain.

- *Real analysis*

The notion of a Cauchy sequence is vital in the study of real analysis in particular. Real analysis is a branch of mathematics that deals with the behaviour and properties of real numbers, sequences and series of real numbers and functions of real numbers.

- *Metric concepts*

Metric concepts are already interwoven in the definition of Cauchy sequences; therefore, it would be logical to generalise the sequence to any metric space [X]. The notion is simple, one needs to replace the absolute value of $|x_m - x_n| < \varepsilon$ with the distance, denoting a metric[8] between both $x_m \ and \ x_n$. Thus, transforming it to:

$$d(x_m - x_n)$$

Following up on that, imagining a metric space (X, d) every sequence is Cauchy when there is a positive integer [N] for every positive real number which is greater than 0. Such that for all positive integers m and n are greater than N. The distance then equals:

$$d(x_m, x_n) > \varepsilon$$

To put it simply the elements become ever closer to each other, in a manner that one would think it converges to a limit in X. This is however not always the case. The Cauchy sequence converges in a space known as completeness.

- *Completeness*

A metric space (X, d), where every Cauchy sequence converges to X is known as a complete sequence. Here Cauchy sequences of rational numbers are used to within the branch of mathematics developing the real number system.

[8] In mathematics, a metric (also known as a distance function) is a function defining the distance between every pair of elements.

- *Completion of categories*

A theory was introduced in 2018, about the definition of Cauchy completion (completeness) of a category. A category is a collection of objects that are linked together with arrows used to prove the existence of an identity arrow for every object. This application of Cauchy sequences involves only a generalised version of the sequence instead of the original sequence.

- *Topology*

A generalised version of Cauchy sequences is used for a topological vector space which researches functional analysis (analysis of functions).

A property that Cauchy sequences and the Fibonacci sequence share is the varied and numbered applications. Cauchy sequences are also not limited to the applications above, many more exist spread out across different branches of mathematics. The main objective of this chapter was to illustrate the many uses of Cauchy sequences and not to give clear detail on each and every use of the sequence, meaning that if a use seems vague or unclear, it is to be expected as only a short description was given, for further detail on each application separate personal research is needed.

Summary of the sequence

Cauchy sequences are complex and limited to mathematical use as well as study, the word Cauchy defines a great number of attributes to the sequence and the properties surrounding it. This sequence may not be vital to the understanding of mathematics; however, it does play an important role in the development of different branches of mathematics. This not well-known sequence was difficult to fully describe without dedicating a lot of this work to it, the notion of the sequence may be simple, the rest on the other and is challenging to fully comprehend.

Farey sequence(s)

Introduction

At the start of the 19th century a mathematical sequence of order n was invented and developed to generate simple and completely reduced fractions lying in the interval [0, 1]. These fractions are arranged in order of increasing size, forming the Farey sequence denoted as F_n, which only contains irreducible, proper, and positive fractions with denominators less than or equal to n. However, the Farey sequence does not exist solely as a manner of sorting fraction on order of their size, there are several studies and research work related with Farey sequences. As well as many studies, the Farey sequence consists of many mathematical properties and is applied to certain areas of mathematics.

Notation

The Farey sequence, with restricted definition, starts with the value 0 represented by a fraction and ends with 1 as its last element also denoted by a fraction.

$$F_n = \left\{\frac{0}{1}, \dots, \frac{1}{1}\right\}$$

This sequence then forms the simplest of all Farey sequences called a Farey sequence of first order. Farey sequence are in theory always finite both ways however, there are an infinite amount of orders of the sequence, meaning the sequence can be denoted in many different forms. For example, the Farey sequence of third order is defined as follows:

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

Given all the elements of the previous order of the sequence are then repeated in the following order, with the added elements from the new order. Here for example, when looking for the fourth order of the Farey sequence, most of the elements are already present in the previous order.

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$

Determining the order of each Farey sequence is a simple matter as the denominators of each fraction are either equal or inferior to n. This leaves only to find the largest denominator, which then equals the order of that Farey sequence. So, for example the Farey sequence:

$$F_n = \left\{\frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}\right\}$$

is a Farey sequence of the eighth order where n is equal to eight.

History

The sequence is named after a British geologist John Farey senior (1766 – 1826), who was supposedly the first person to discover the properties of rational numbers, upon which the sequence consists of. Farey was believed to have said that each new element in a Farey sequence expansion is the median of its neighbours, for which he offered no proof. In 1816, Farey published a four paragraphed article in a magazine, here he described the property and the sequence, later he followed to state the fifth order of the sequence. He ends the article by asking himself if he is the first to have noticed the property.

Missing however, was the proof that confirmed Farey's theory, this came later that year and was provided by French mathematician Augustin-Louis Cauchy (who also developed the Cauchy sequences named in his honour), proving Farey's observation. At the time it was believed that Farey was the first person to notice the property, and the sequence was therefore named after him. It was later discovered that Farey was in fact not the first to discover the property, but that Charles Haros had noticed the property well before Farey in 1802, 14 years beforehand. However, Haros never received any credit for his discovery. Due to this controversial fact Farey is not regarded favourably within the mathematical community, as he was immortalised because he did not understand a relation that was understood with proof, 14 years beforehand.

Properties

Here, most of the basic properties of the sequence will be listed, which will be followed by two core properties that shape the sequence.

Basic properties

- for a sufficiently large value of n, the approximate number of fractions in F_n is given by

$$\frac{3n^2}{\pi^2}$$

- given a real number x, there is always a fraction close to it $\frac{a}{b}$ [element of a Farey sequence], which:

$$\left|x - \frac{a}{b}\right| \leq \frac{1}{(b(n+1))}$$

Farey neighbours

Farey neighbours is the name given to bordering elements [fractions] in a Farey sequence, these are known as a Farey Pair and have interesting properties.

- For any two fractions succeeding each other $\frac{a}{b}$ and $\frac{c}{d}$ in the Farey sequence:

$$b + d \geq n + 1$$

- For any two fractions succeeding each other $\frac{a}{b}$ and $\frac{c}{d}$ in the Farey sequence, where $\frac{a}{b} < \frac{c}{d}$

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$$

Proof:

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}$$

Meaning:

$$bc - ad = 1$$

Example F_7:

$$\frac{2}{5}, \frac{3}{7} \rightarrow \frac{3}{7} - \frac{2}{5} = \frac{1}{35}$$

$$15 - 14 = 1$$

- For positive integers a, b, c and d, where a < b and c < d, $\frac{a}{b}$ and $\frac{c}{d}$ will border each other in the sequence of a maximum order of (b, d)

- For any three fractions succeeding each other $\frac{a}{b}, \frac{p}{q}\ and\ \frac{c}{d}$ in the F_n , where $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$

$$\frac{p}{q} = mediant\ of\ \frac{a}{b}\ and\ \frac{c}{d}$$

Meaning:

$$\frac{p}{q} = \frac{a+c}{b+d}$$

Example F_8:

$$\frac{3}{7}, \frac{1}{2}, \frac{4}{7} \rightarrow \frac{1}{2} = \frac{3+4}{7+7}$$

- Therefore, if $\frac{a}{b}\ and\ \frac{c}{d}$ border in a Farey sequence, the first elements that splits them, as the order of the sequence is incremented, is :

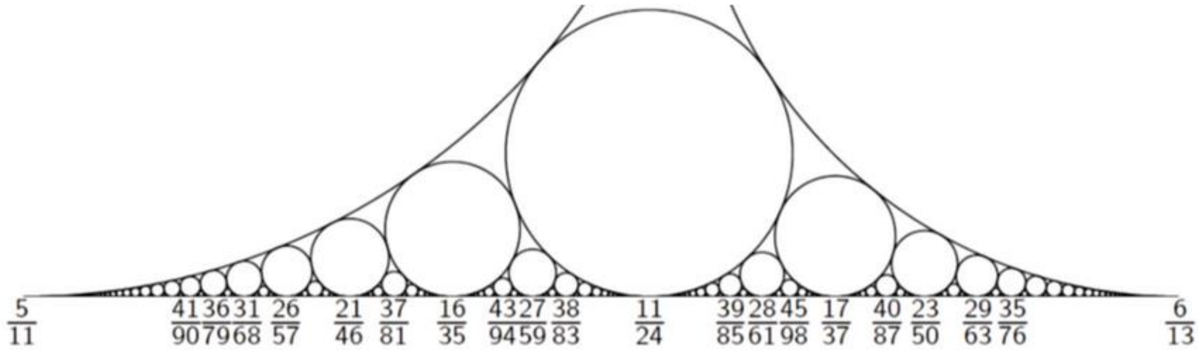
$$\frac{a+c}{b+d}$$

- The amount of Farey pairs in a sequence can be determined by

$$2|F_n| - 3$$

Ford circles

Farey sequences share an unusual property with Ford circles, this connection is a notable occurrence in the research of Farey sequences. Ford circles are circles with set measures of creation, they are constructed by drawing a circle with a diameter of $\frac{1}{q^2}$, directly above an irreducible fraction of the Farey sequence $\frac{p}{q}$. The phenomenon looks like this:

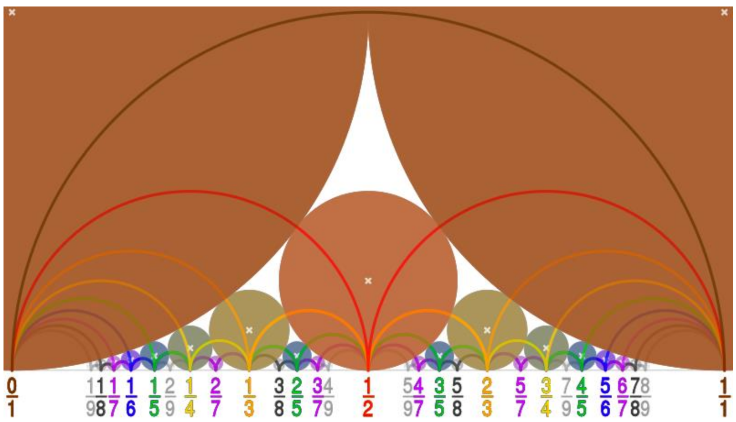


Observable is that for every fraction there is a Ford Circle [denoted: $C\left[\frac{2}{5}\right]$].

The second observation that can be made is that the circles do not overlap each other. This particularity means that two Ford circles for different fractions are either disjoint[9] or tangent[10]. Given that $\frac{p}{q}$ is a fraction with a value between 0 and 1, the circles which are tangent to its Ford circle [denoted: $C\left[\frac{2}{5}\right]$] are exactly the same circles of the neighbouring fractions to $\frac{p}{q}$ in the Farey sequence. To conclude, here is an example

$$C\left[\frac{2}{5}\right] \textit{ is tangent to } C\left[\frac{1}{2}\right], \left[\frac{1}{3}\right], \left[\frac{3}{7}\right], \left[\frac{3}{8}\right] \textit{ and so on}$$

Observable through this image:



[9] Disjoint circles do not share a common element/fraction

[10] Tangent represents the fact that the circles "only just touch each other"

Applications and summary

Applications

The applications of Farey sequences are severely limited compared to the two previous sequences, the sequences entail only a few uses in specific areas of mathematics and physics. Here is a list of a few of the uses of Cauchy sequences, the domain will be stated, followed by a short explanation about the use of the sequence.

In mathematics, Farey sequences can be utilised to identify rational approximations of irrational numbers. An obvious example of this is the calculation of fraction expansion, which unsurprisingly uses Farey sequences. Furthermore, Farey sequences facilitate the characterisation of the computational complexity of square-celled grids, which is needed in various studies within any-angle path planning. Finally, in physics, Farey sequences provide a simplified and efficient method to compute resonance locations in both 1- and 2D.

Summary

The Farey sequence provides an attractive way of sorting fractions by size; however, Farey sequences consist of much more than solely a way of sorting fractions. Although the list of applications is limited compared to other sequences, it offers an intriguing collection of properties, which do not immediately become apparent. The graphical representations and Ford circles furnish images containing a certain amount of beauty, which makes it stand out from other sequences. Taking this into consideration, Farey sequences are my favoured sequences due to their elegant notation and promising properties. Along with a controversial piece of history, Farey sequences are visually pleasing and interesting.

Conclusion

Personally, I set out at the start of this year to attain a finalised document, which fully explains what a sequence is made up of and each different component explained in full detail. After months of research I am pleased to announce that I have achieved this goal. My project started out a bit shaky, however, as my understanding of sequences grew, the separate elements came together to form a complete sequence.

Furthermore, I aimed to obtain a greater understanding of sequences and sequencing, as well as a more rounded off level of comprehension of mathematics in general. This has been particularly helpful, as it has fuelled my curiosity to venture further into the field of mathematics. After having completed this work, and after doing this research, I no longer fear the seemingly huge field of mathematics, understanding a portion of the field at one time is possible.

Another promising benefit of this work is that is has helped decide what career path I wish to follow after completing secondary school. The answer was clear after reflecting upon my introduction, I set out to not only increase my knowledge on the subject but to also share it with my classmates. This prompted me to consider a teaching career, combined with my strengths that lie in mathematics and science, the result being a qualified teacher of mathematics at secondary school level.

After having reread my work and reflected upon my experiences writing and researching this topic, I conclude that sequences and sequencing might not appear as a subject of interest. However, it has proven the exact opposite to me, I enjoyed researching the topic, as well as adding the personal touch with the individual sequences to my work. A part of my research was not added to this work, as it was merely personal endeavour, so in conclusion I would recommend this topic, for its different and generally unheard-of aspects, which will not fail to surprise you.

Finally, through this work I am able to conclude that sequences, their properties and applications consist of much more than many might think. Their diversity and variation make them a special part of the field of mathematics. Mathematical sequences serve as an interesting entry point into more complicated maths, as there are many connections to other fields, helping to understand a greater deal of mathematics than just sequences and sequencing.

I have discovered and determined that the notion of sequences is not merely confined to mathematics, but that sequences are used across a wide range of different domains such as science, geography and photography among others. Sequences make up a vital part of our daily routines and lives, due to their important roles in society, for example time, calendars, etc. What makes this subject great is that, the concept may be simple, however the subject branches off in all directions, offering more difficult ideas for the people who are interested. An underrated and mostly unheard-of field in mathematics, brings with it excellent and interesting content.

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